

LIMITING DISTRIBUTION OF THE MAXIMAL DISTANCE BETWEEN RANDOM POINTS ON A CIRCLE: A MOMENTS APPROACH

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ABSTRACT. Motivated by the problem of computing the distribution of the largest distance d_{\max} between n random points on a circle we derive an explicit formula for the moments of the maximal component of a random vector following a Dirichlet distribution with concentration parameters $(1, \dots, 1)$. We use this result to give a new proof of the fact that the law of $n d_{\max} - \log n$ converges to a Gumbel distribution as n tends to infinity.

1. INTRODUCTION

For a positive integer n , we denote by X_1, \dots, X_n a collection of n independent, standard uniform random variables, which we interpret as locations of points on a circle with perimeter one. By d_i , $i = 1, \dots, n$, we denote the distances (in arc length) between adjacent points, that is $d_i = X_{i+1} - X_i$, where the index is taken modulo n . Alternatively, the distances d_i can be interpreted as the lengths of the pieces of a randomly broken stick of length one [Holst \[1980\]](#). A detailed understanding of their properties is of importance in some aspects of non-parametric statistics [Wilks \[1962\]](#). The set of distances is also interesting from a purely probabilistic point of view because the smallest, typical, and largest distances show quite markedly different behaviour as the number of points tend to infinity. It is known [[David and Nagaraja, 2003](#), Problem 6.4.2] that the expected size of the k^{th} -largest gap, $k = 1, \dots, n$, is given by $n^{-1} \sum_{j=k}^n 1/j$. In particular, the smallest gap d_{\min} is of order $1/n^2$, whereas the largest gap d_{\max} is of order $H_n/n \sim \log n/n$, where $H_n = \sum_{j=1}^n 1/j$ denotes the n^{th} harmonic number, and $a_n \sim b_n$ if and only if $a_n/b_n \rightarrow 1$. An easy calculation shows that $n^2 d_{\min}$ converges in distribution to an exponential random variable with parameter one. In this short note, we are concerned with the limiting distribution of a suitably scaled and centred version of d_{\max} . Using the observation that the n -tuple (d_i) of distances follows a Dirichlet distribution with parameters $(1, \dots, 1)$ it can be deduced from [[Bose et al., 2008](#), Corollary 3.1.] that $n d_{\max} - \log n$ converges in law to a Gumbel distribution.

In the following we provide an alternative, combinatorial proof of that result, bypassing arguments from extreme value theory in the spirit of [Gnedenko \[1943\]](#) and [Leadbetter et al. \[1983\]](#); we derive, for the first time, an explicit formula for the moments of a $\text{Dir}(1, \dots, 1)$ distribution and compute their limits as n tends to infinity. This allows us to identify the limiting distribution in Section 3.

2. COMPUTATION OF THE MOMENTS OF d_{\max}

By [[David and Nagaraja, 2003](#), Eq. (6.4.4)], the distribution function of the largest gap is given by

$$\mathbb{P}(d_{\max} \leq x) = 1 - n(1-x)^{n-1} + \binom{n}{2}(1-2x)^{n-1} - \dots + (-1)^k \binom{n}{k}(1-kx)^{n-1} + \dots,$$

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where the sum continues as long as $kx \leq 1$. In particular, after differentiating with respect to x and observing that $d_{\max} \geq 1/n$, the m^{th} moment of the largest spacing is given by

$$\mathbb{E}[(d_{\max})^m] = (n-1) \sum_{\nu=1}^{n-1} \int_{1/(\nu+1)}^{1/\nu} \sum_{k=1}^{\nu} \binom{n}{k} (-1)^{k+1} k x^m (1-kx)^{n-2} dx. \quad (2.1)$$

In the following we evaluate this expression in closed form. Since the order in which the summations and integration are carried out is inconsequential, we consider the integrals in Eq. (2.1) first.

Lemma 1. *For positive integers n , $k < n$, $\nu < n$ and m , the following holds.*

$$\int_{1/(\nu+1)}^{1/\nu} x^m (1-kx)^{n-2} dx = \sum_{\mu=0}^m \frac{1}{k^\mu} \frac{m!(n-2)!}{(m-\mu)!(n+\mu-2)!} T_{n+\mu,k,\nu,m-\mu}, \quad (2.2)$$

where

$$T_{n+\mu,k,\nu,m-\mu} = \frac{1}{k(n+\mu-1)} \left(\frac{(\nu+1-k)^{n+\mu-1}}{(\nu+1)^{n+m-1}} - \frac{(\nu-k)^{n+\mu-1}}{\nu^{n+m-1}} \right). \quad (2.3)$$

Proof. The result is obtained by m -fold integration by parts. \square

After changing the order of summations, this result can be used to perform the ν -sum in Eq. (2.1).

Lemma 2. *For positive integers n , $k < n$ and m , the following holds.*

$$\sum_{\nu=k}^{n-1} \int_{1/(\nu+1)}^{1/\nu} x^m (1-kx)^{n-2} dx = \sum_{\mu=0}^m \frac{1}{k^{\mu+1}} \frac{m!(n-2)!}{(m-\mu)!(n+\mu-1)!} \frac{(n-k)^{n+\mu-1}}{n^{n+m-1}}. \quad (2.4)$$

Proof. After plugging in Eq. (2.2) and interchanging the order of summation the sum over k is seen to be telescoping, which gives the result. \square

We also need the following binomial identities whose easy proofs are left to the reader.

Lemma 3. *For positive integers n , m and $s \leq m$, the following identities hold.*

$$\sum_{k=1}^n \binom{n}{k} (-1)^{k+1} \frac{k}{n+1-k} = (-1)^{n+1}; \quad (2.5)$$

$$\sum_{k=1}^n k^s (-1)^{k+1} \binom{n}{k} = 0; \quad (2.6)$$

$$\sum_{\mu=s}^m \frac{(-1)^\mu}{(m-\mu)!(n+\mu-1)!} \binom{n+\mu-1}{\mu-s} = \delta_{s,m} \frac{(-1)^m}{(n+m-1)!}. \quad (2.7)$$

The following result records a link between raw moments and cumulants of a random variable and is used repeatedly in the sequel.

Lemma 4. *For a positive integer m and real numbers x_1, \dots, x_m , the quantity*

$$\sum_{\substack{r_1+2r_2+\dots+mr_m=m \\ r_i \in \mathbb{N}_0}} \frac{m!}{r_1! 1^{r_1} \dots r_m! m^{r_m}} x_1^{r_1} \dots x_m^{r_m} \quad (2.8)$$

can be interpreted as $m! [y^m] \exp \left\{ \sum_{r=1}^m x_r y^r / r \right\}$, where $[y^m] f(y)$ denotes the coefficient of y^m in the formal power series $f(y)$. In particular, the m^{th} cumulant of a random variable with m^{th} raw moment given by Eq. (2.8) is equal to $\kappa_m = (m-1)! x_m$.

Proof. The first claim is proved by rearranging terms. It implies that the moment generating function of a random variable with raw moments μ'_m given by Eq. (2.8) is

$$M(t) = 1 + \mu'_1 t + \frac{\mu'_2}{2!} t^2 + \frac{\mu'_3}{3!} t^3 + \dots = \exp \left\{ \sum_{r=1}^{\infty} \frac{x_r}{r} t^r \right\}.$$

Its cumulants are thus easily computed from the cumulant generating function $K(t) = \log M(t)$ as

$$\kappa_m = \left. \frac{d^m}{dt^m} K(t) \right|_{t=0} = \left. \frac{d^m}{dt^m} \sum_{r=1}^{\infty} \frac{x_r}{r} t^r \right|_{t=0} = (m-1)! x_m. \quad \square$$

For the statement of the following auxiliary result, which we could not find proved in the literature, we introduce the notation

$$\mathcal{H}_{n,s} := \frac{1}{s!} \sum_{\mathbf{r}(s)} \binom{s}{\mathbf{r}} H_{n,1}^{r_1} \cdots H_{n,s}^{r_s} := \sum_{\substack{r_1+2r_2+\dots+sr_s=s \\ r_i \in \mathbb{N}_0}} \frac{1}{r_1! 1^{r_1} \cdots r_s! s^{r_s}} H_{n,1}^{r_1} \cdots H_{n,s}^{r_s},$$

where $\sum_{\mathbf{r}(s)}$ denotes a sum over integer partitions of s and $H_{n,r} = \sum_{j=1}^n 1/j^r$ is the n^{th} harmonic number of order r . The numbers $\mathcal{H}_{n,s}$, whose expression is intimately related to the Bell polynomials Bell [1927/28], figure prominently in our expression for the moments of d_{\max} .

Lemma 5. *For any positive integers n and s the following holds.*

$$\sum_{k=1}^n k^{-s} \binom{n}{k} (-1)^{k+1} = \mathcal{H}_{n,s}. \quad (2.9)$$

Proof. The proof proceeds by induction on n . By Lemma 4 it suffices to show that the left side of Eq. (2.9) equals the coefficient of y^s in $\exp \left\{ \sum_{r=1}^s H_{n,r} y^r / r \right\}$. The claim is true for $n = 1$, when both sides equal one. For the induction step we assume the validity of the statement up to n and compute

$$\begin{aligned} [y^s] \exp \left[\sum_{r=1}^s \frac{H_{n+1,r}}{r} y^r \right] &= [y^s] \exp \left\{ \sum_{r=1}^s \frac{H_{n,r}}{r} y^r + \sum_{r=1}^{\infty} \frac{1}{r} \left(\frac{y}{n+1} \right)^r \right\} \\ &= \sum_{\sigma=0}^s \left([y^{\sigma}] \exp \left\{ \sum_{r=1}^s \frac{H_{n,r}}{r} y^r \right\} \right) \left([y^{s-\sigma}] \frac{n+1}{n+1-y} \right) \\ &= \sum_{\sigma=0}^s \sum_{k=1}^n k^{-\sigma} \binom{n}{k} (-1)^{k+1} \frac{1}{(n+1)^{s-\sigma}} \\ &= \sum_{k=1}^n \binom{n}{k} (-1)^{k+1} \left[k^{-s} \frac{n+1}{n+1-k} - (n+1)^{-s} \frac{k}{n+1-k} \right] \\ &= \sum_{k=1}^{n+1} k^{-s} \binom{n+1}{k} (-1)^{k+1}. \quad \square \end{aligned}$$

To obtain the last line we have used the identity (2.5).

The main result of this section can now be proved.

Theorem 1. For positive integers n and m , the m^{th} moment of d_{\max} is given by

$$\mathbb{E}[(d_{\max})^m] = \frac{(n-1)!}{(n+m-1)!} \sum_{\substack{r_1+2r_2+\dots+mr_m=m \\ r_i \in \mathbb{N}_0}} \frac{m!}{r_1! 1^{r_1} r_2! 2^{r_2} \dots r_m! m^{r_m}} H_{n,1}^{r_1} H_{n,2}^{r_2} \dots H_{n,m}^{r_m}. \quad (2.10)$$

Proof. Combining Eq. (2.1) and Lemma 2 we can write

$$\mathbb{E}[(d_{\max})^m] = (n-1) \sum_{k=1}^{n-1} \binom{n}{k} (-1)^{k+1} \sum_{\mu=0}^m \frac{1}{k^\mu} \frac{m!(n-2)!}{(m-\mu)!(n+\mu-1)!} \frac{(n-k)^{n+\mu-1}}{n^{n+m-1}}.$$

We use the binomial theorem to expand $(n-k)^{n+\mu-1}$ and change the order of summation to obtain

$$\mathbb{E}[(d_{\max})^m] = \frac{(n-1)!}{n^m} \sum_{\mu=0}^m \frac{m!}{(m-\mu)!(n+\mu-1)!} \sum_{s=0}^{n+\mu-1} (-1)^s n^{\mu-s} \binom{n+\mu-1}{s} \sum_{k=1}^{n-1} k^{s-\mu} \binom{n}{k} (-1)^{k+1}.$$

Splitting the sum according to the sign of the exponent $s-\mu$ of k , adjusting the summation index s , and using Lemma 5 as well as Eq. (2.6), we obtain

$$\begin{aligned} \mathbb{E}[(d_{\max})^m] &= \frac{(n-1)!}{n^m} \sum_{\mu=0}^m \frac{m!}{(m-\mu)!(n+\mu-1)!} \left[\sum_{s=1}^{\mu} (-1)^{\mu-s} n^s \binom{n+\mu-1}{\mu-s} \sum_{k=1}^{n-1} k^{-s} \binom{n}{k} (-1)^{k+1} + \right. \\ &\quad \left. + \sum_{s=0}^{n-1} (-1)^{\mu+s} n^{-s} \binom{n+\mu-1}{\mu+s} \sum_{k=1}^{n-1} k^s (-1)^{k+1} \binom{n}{k} \right] \\ &= \frac{(n-1)!}{n^m} \sum_{\mu=0}^m \frac{m!}{(m-\mu)!(n+\mu-1)!} \left[(-1)^\mu \sum_{s=1}^{\mu} (-1)^s n^s \binom{n+\mu-1}{\mu-s} \mathcal{H}_{n,s} + \right. \\ &\quad \left. + (-1)^{n+\mu} \sum_{s=1}^{\mu} (-1)^s \binom{n+\mu-1}{\mu-s} + (-1)^{n+\mu} \sum_{s=0}^{n-1} (-1)^s \binom{n+\mu-1}{\mu+s} \right]. \end{aligned}$$

The last two terms are equal to $\pm(n+\mu-2)!/[(n-1)!(\mu-1)!]$ and cancel each other. Interchanging the order of summation and using Eq. (2.7) we finally obtain

$$\mathbb{E}[(d_{\max})^m] = \frac{(n-1)!}{n^m} \sum_{s=1}^m \delta_{s,m} \frac{m!(-1)^m}{(n+m-1)!} (-1)^s n^s \mathcal{H}_{n,s} = \frac{(n-1)!m!}{(n+m-1)!} \mathcal{H}_{n,m}. \quad \square$$

3. IDENTIFICATION OF THE LIMITING DISTRIBUTION

In this section we leave the finite setting and explore the asymptotic behaviour of the largest spacing d_{\max} . We use $o(1)$ to denote a term that converges to zero as n tends to infinity. The first lemma shows that simply rescaling d_{\max} by the inverse of its expected size does not lead to an interesting limit.

Lemma 6. The sequence $\frac{n}{\log n} d_{\max}$ converges to one in distribution.

Proof. Using Theorem 1 one sees that all moments converge to one. Since the Dirac mass at one is the only measure with all moments equal to one, the claim follows. \square

Since $\mathbb{E}\left[\frac{n}{\log n} d_{\max} - 1\right] = [\gamma + o(1)] / \log n$, it is natural to consider $\log n \left(\frac{n}{\log n} d_{\max} - 1\right)$ next. This scaling turns out to be correct.

Theorem 2. *The moments of the centred and scaled maximal distance $n d_{\max} - \log n$ satisfy*

$$\mathbb{E}[(n d_{\max} - \log n)^m] \xrightarrow{n \rightarrow \infty} \mu_m := \sum_{\mathbf{r}(m)} \binom{m}{\mathbf{r}} \gamma^{r_1} \zeta(2)^{r_2} \cdots \zeta(m)^{r_m}, \quad (3.1)$$

where γ is the Euler–Mascheroni constant and $\zeta(\cdot)$ denotes the Riemann ζ -function. In particular, the m^{th} cumulant of $n d_{\max} - \log n$ converges to γ for $m = 1$, and to $(m - 1)! \zeta(m)$ for $m \geq 2$.

Proof. The binomial theorem implies that the left side of Eq. (3.1) can be written as

$$\sum_{k=0}^m (\log n)^{m-k} n^k \binom{m}{k} (-1)^{m-k} \frac{(n-1)!}{(n+k-1)!} \sum_{\mathbf{r}} \binom{k}{\mathbf{r}} H_{n,1}^{r_1} \cdots H_{n,k}^{r_k}.$$

Since $(n-1)!/(n+k-1)! \sim n^{-k}$, it follows that, in the limit, this sum can be interpreted as the m^{th} moment of $\tilde{d}_{\max} - \log n$, where \tilde{d}_{\max} is a random variable with raw moments $\sum_{\mathbf{r}} \binom{m}{\mathbf{r}} H_{n,1}^{r_1} \cdots H_{n,m}^{r_m}$. For such a random variable, however, the cumulants are easily computed using Lemma 4, namely

$$\kappa_{n,1} = H_{1,n} = \log n + \gamma + o(1), \quad \text{and} \quad \kappa_{n,m} = (m-1)! H_{n,m} = (m-1)! [\zeta(m) + o(1)], \quad m \geq 2.$$

Using the fact that the first cumulant is shift-equivariant and that higher cumulants are shift-invariant we find that the cumulants of $\tilde{d}_{\max} - \log n$, and thus also the cumulants of $n d_{\max} - \log n$, converge to

$$\kappa_1 = \gamma, \quad \text{and} \quad \kappa_m = (m-1)! \zeta(m), \quad m \geq 2.$$

This convergence of cumulants implies convergence of moments to the limit given in the statement of the theorem because each moment is a continuous function of finitely many cumulants. \square

We can now use this information to identify the limiting distribution of the maximal spacing between random points on a circle.

Corollary 1. *The rescaled and centred maximal distance $n d_{\max} - \log n$ converges in law to a standard Gumbel distribution with location parameter zero and scale one.*

Proof. It is well known that the limiting cumulants of $n d_{\max} - \log n$ obtained in Theorem 2 are exactly those of a standard Gumbel distribution. It thus suffices to show that these cumulants, or equivalently the moments (μ_m) determine a unique probability distribution. This, however, follows from [Durrett, 2010, Theorem 3.3.11.] if it can be shown that the sequence $(\mu_{2m}^{1/2m}/2m)$ remains bounded. Following Apostol [1976] to bound the number of partitions of $2m$ by $\exp\{c\sqrt{2m}\}$, where $c = 2\sqrt{\zeta(2)}$, and further bounding $\gamma < 2$, $\zeta(r) < 2$, it follows that

$$\frac{1}{2m} \mu_{2m}^{1/2m} < \frac{1}{2m} \left(\exp\{c\sqrt{2m}\} (2m)! 2^{2m} \right)^{1/2m} \xrightarrow{m \rightarrow \infty} \frac{2}{e} < \infty.$$

This completes the proof of the corollary. \square

REFERENCES

- T. M. Apostol. *Introduction to analytic number theory*. Springer-Verlag, New York, 1976.
- E. T. Bell. Partition polynomials. *Ann. of Math. (2)*, 29(1-4):38–46, 1927/28.
- A. Bose, A. Dasgupta, and K. Maulik. Maxima of Dirichlet and triangular arrays of gamma variables. *Statist. Probab. Lett.*, 78(16):2811–2820, 2008.
- H. A. David and H. N. Nagaraja. *Order statistics*. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2003.
- R. Durrett. *Probability: theory and examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, fourth edition, 2010.

- B. Gnedenko. Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. of Math. (2)*, 44:423–453, 1943.
- L. Holst. On the lengths of the pieces of a stick broken at random. *J. Appl. Probab.*, 17(3):623–634, 1980.
- M. R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer-Verlag, New York, 1983.
- S. S. Wilks. *Mathematical statistics*. A Wiley Publication in Mathematical Statistics. John Wiley & Sons Inc., New York, 1962.

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